INVESTIGATION OF THE DEFORMATION OF ELASTIC MEMBRANES CONSTRAINED BY RESTRICTIONS (ON DISPLACEMENTS) BY THE METHOD OF DYNAMIC PROGRAMING

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V. A. BARANENKO and Iu. M. POCHTMAN
(Dnepropetrovsk)
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Contact problems on the deflection shapes of elastic membranes (serving as elements of the type of flooring, sheathing, etc. in real structures) in the presence of restrictions on the deflection are considered. In a variational formulation the mentioned problems reduce to finding functions which minimize some functional for given boundary conditions, and simultaneously satisfy restrictions of inequality type. One of the modern mathematical, optimal-control methods is used for numerical solution of the problems on an electronic computer, namely, dynamic programing [1], based on the "optimality principle" of R. Bellman as applied to the multistep process of finding the solution.

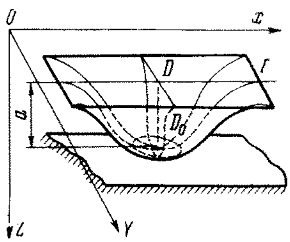

Fig. 1

Let us consider a rectangular elastic membrane having the constant tension F and loaded by an arbitrary transverse loading $q(x,y)$. The membrane is clamped along the contour Γ, which is the boundary of some domain D (Fig. 1) in the Oxy plane. The membrane deformations are restricted from below by an absolutely rigid flat wall parallel to the undeformed membrane contour and standing a distance a off. The problem of finding the membrane deflections $w(x,y)$ reduces to integrating a harmonic equation

$$\nabla^2 w = -\frac{q(x, y)}{F} \quad \text{in } D - D_0 \tag{1}$$

(outside the domain of contact D_0 between the membrane and the wall), where ∇^2 is the Laplace operator.

Moreover, the condition $w(x, y) = a$ should be satisfied in the contact domain D_0 Utilizing the theorem of a minimum functional [2], and introducing the dimensionless variables

$$x' = \frac{x}{l}, \quad y' = \frac{y}{l}, \quad J = \frac{J}{l^2}, \quad w' = \frac{wF}{ql^2}, \quad a' = \frac{aF}{ql^2}$$

we replace the solution of this boundary value problem by the solution of a variational problem to determine a function $w(x, y)$ which will minimize a functional of the membrane's potential energy (the primes are henceforth omitted conventionally)

$$J = \int_D \left\{ \frac{1}{2} \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] + q(x, y) w \right\} dx\, dy \tag{2}$$

and satisfy the boundary condition $w|_\Gamma = 0$ and the inequality

$$w(x, y) \leqslant a \quad \text{in } D \tag{3}$$

The presence of the restriction (3) considerably complicates the solution of the problem under consideration, and what is particularly important, does not permit reliance on classical elasticity theory methods for the investigation.

It will be show below that the method of dynamic programing [3] turns out to be very effective for the analysis and numerical solution of this kind of two-dimensional problem on an electronic computer (with a large volume memory) if these problems are interpreted as multistep processes of finding the solution. Let us note that a solution was given by the dynamic programing method in [4] for the contact problem when the desired function was a function of one variable (elastic bending of a rod).

In order to solve the posed problem by the method of dynamic programing in discrete form, let us divide the rectangular domain D into a mesh with nodes at the vertices (x_i, y_j) $i = 0, 1, 2..., m$; $j = 0, 1, 2..., n$. We shall calculate the values of the function $w(x, y)$ and its derivatives only at the obtained nodes, i.e.

$$w(x_i, y_j) = w_{ij}, \quad \frac{\partial w}{\partial x} = \frac{w_{i+1,j} - w_{ij}}{\Delta x}, \quad \frac{\partial w}{\partial y} = \frac{w_{i,j+1} - w_{ij}}{\Delta y}$$

In this case the problem of minimizing the integral (2) is replaced by the following: to minimize the target function

$$J = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \left\{ \frac{1}{2} \left[\left(\frac{w_{i+1,j} - w_{ij}}{\Delta x} \right)^2 + \left(\frac{w_{i,j+1} - w_{ij}}{\Delta y} \right)^2 \right] - q_{ij} w_{ij} \right\} \Delta x \, \Delta y \tag{4}$$

under the conditions that

$$w_{ij|\Gamma} = 0, \quad w(x_i y_j) \leqslant a \text{ in } D, \quad a \geqslant 0 \tag{5}$$

Let $F_k(c_1, ..., c_{m-1})$ be the minimum of J in all $w_{ij} \neq w_{ik}$ under the condition that the process starts at the time k from the state $\{c_1 ..., c_{m-1}\}$ and continues to $k = (n-1)$ stages at an optimal strategy, i.e.,

$$F_k(c_1, \ldots, c_{m-1}) = \min \sum_{i=0}^{m-1} \sum_{j=k}^{n-1} \left\{ \frac{1}{2} \left[\left(\frac{w_{i+1,j} - w_{ij}}{\Delta x} \right)^2 + \left(\frac{w_{i,j+1} - w_{ij}}{\Delta y} \right)^2 \right] - q_{ij} w_{ij} \right\} \Delta x \, \Delta y \tag{6}$$

where $w_{ik} = c_i$ and the boundary conditions (5) are satisfied. Then, according to the optimality principle of dynamic programing [3], the functional equations for the posed problem are written as

$$F_k(c_1, \ldots, c_{m-1}) = \min \left\{ \sum_{i=0}^{m-1} \left(\frac{1}{2} \left[\left(\frac{c_{i+1} - c_i}{\Delta x} \right)^2 + \right. \right. \right. \tag{7}$$

$$\left. \left. \left. + \left(\frac{w_{i,k+1} - c_i}{\Delta y} \right)^2 \right] - q_{ik} c_i \right) \Delta x \, \Delta y + F_{k+1}(w_{1,k+1}, w_{2,k+1}, \ldots, w_{m-1,k+1}) \right\} \quad (k = 0, 1, \ldots n-2)$$

Here w_{0k}, w_{mk}, $w_{0,k+1}$, $w_{m,k+1}$ are known according to (5). For F_{n+1} we have

$$F_{n-1}(c_1, \ldots, c_{m-1}) = \sum_{i=0}^{m-1} \left\{ \frac{1}{2} \left[\left(\frac{c_{i+1} - c_i}{\Delta x} \right)^2 + \left(\frac{w_{in} - c_i}{\Delta y} \right)^2 \right] - q_{in} c_i \right\} \tag{8}$$

for known w_{in}, $w_{0,n-1}$, $w_{m,n-1}$.

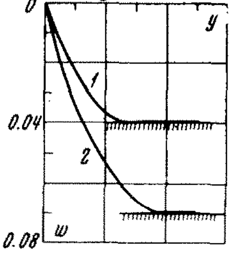

Fig. 2

Executing the algorithm (7), (8) on an electronic computer, we find values of the target function (4) while taking account of (5), and also the desired values of $w(x,y)$ at discrete points of the mesh domain x_i, y_j.

A square membrane with sides $l = 1$, loaded by a uniformly distributed loading of intensity $q = 1$ and restricted by planes $a = 0.04$ and $a = 0.07$, was computed on an electronic computer by means of the algorithm presented above as a numerical illustration. The domain under investigation was approximated by a rectangular mesh with $m = 4$ and $n = 8$. In this case (7) and (8) are written as

$$F_k(c_1, c_2, c_3) = \min_{w_{i,k+1}} \left\{ \sum_{i=0}^{3} \left(\frac{1}{2} \left[\left(\frac{c_{i+1} + c_i}{\Delta x} \right)^2 + \left(\frac{w_{i,k+1} - c_i}{\Delta y} \right)^2 \right] - c_i \right) \Delta x \, \Delta y + \right.$$

$$\left. + F_{k+1}(w_{1,k+1}, w_{2,k+1}, w_{3,k+1}) \right\} \quad (k = 0, 1, \ldots 6)$$

$$F_7(c_1, c_2, c_3) = \sum_{i=0}^{3} \left(\frac{1}{2} \left[\left(\frac{c_{i+1} - c_i}{\Delta x} \right)^2 + \left(\frac{w_{i8} - c_i}{\Delta y} \right)^2 - c_i \right) \Delta x \, \Delta y$$

Results of the calculations are presented in Fig. 2. Membrane deflections at the section $x = 0.5$ are shown for $a = 0.04$ and $a = 0.07$ by curves *1* and *2* , respectively. The value of the functional at $a = 0.07$ turned out to be $J = -0.0151$. The results obtained are in good agreement with the results in [5], where the solution of an analogous problem was performed by the method of local variations.

In conclusion, let us note that dynamic programing can be applied to solve such a class of two-dimensional problems even for restrictions of more general type on the deformation.

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SPLITTING OF AN INFINITE ELASTIC WEDGE

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B. I. SMETANIN
(Rostov-on-Don)

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The problem of splitting an infinite elastic wedge with a thin perfectly rigid smooth plate is considered. The plate is driven in along the bisectrix of the wedge angle and a slot forms in front of it, when $a \leqslant r \leqslant b$. The wedge faces are either free or hinged.

Formulas defining the form of the slot surface and the normal stress intensity coefficient are obtained. Effective asymptotic methods developed in [1] as well as the mathematical apparatus of the Wiener-Hopf method [2] are employed in the course of solution.

1. Statement of the problem. Solution of the problem by approximating the function L. Let a thin perfectly rigid smooth plate of constant